

HESSIAN EQUATIONS ON CLOSED HERMITIAN MANIFOLDS

DEKAI ZHANG

ABSTRACT. In this paper, using the technical tools in [14], we solve the complex Hessian equation on closed Hermitian manifolds, which generalizes the the Kähler case results in [4] and [3].

1. INTRODUCTION

Let (M, g) be a compact Hermitian manifold of complex dimension $n \geq 2$, and ω be the corresponding Hermitian form. In local coordinates, we write ω as

$$\omega = \sqrt{-1} \sum_{i,j=1}^n g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

In this paper, we consider the following Hessian equation on closed Hermitian manifolds

$$(1.1) \quad \begin{cases} C_n^k \omega_u^k \wedge \omega^{n-k} = e^f \omega^n, & \sup_M u = 0 \\ \omega_u = \omega + \sqrt{-1} \partial \bar{\partial} u \in \Gamma_k(M), \end{cases}$$

where $\Gamma_k(M)$ is a convex cone defined in (2.2) in section 2.

When $k = n$, the condition $\omega_u \in \Gamma_k(M)$ is equivalent to $\omega_u > 0$. Equation (1.1) becomes the following Monge-Ampere equation

$$(1.2) \quad \omega_u^n = e^f \omega^n, \quad \sup_M u = 0.$$

In addition, when (M, ω) is a Kähler manifold, i.e., $d\omega = 0$, Yau [16] solved the equation (1.2) now known as Calabi-Yau theorem. For general Hermitian manifolds, the equation (1.1) has been solved by Cherrier [1] in the case of dimensions 2 and Tosatti-Weinkove [11] for arbitrary dimension. For further background, we refer the reader to [10], [11], [5], [17] and the references therein.

When $2 \leq k \leq n - 1$, ω_u may not be positive, the analysis becomes more complicated. Suppose that (M, ω) is a Kähler manifold and $\omega_u \in \Gamma_k(M)$ which is defined in section 2, Hou-Ma-Wu [4] proved the following second order estimates of the equation (1.1)

$$(1.3) \quad \max |\partial \bar{\partial} u|_g \leq C(1 + \max |\nabla u|_g^2).$$

They also pointed out in their paper that (1.3) may be adapted to the blowing up analysis. Later on, Dinew-Kolodziej [3] obtained the gradient estimate by (1.3). Thus equation (1.1) can be solved on Kähler manifolds under the compatible condition

$$\int_M e^f \omega^n = \int_M \omega^n.$$

Tosatti-Weinkove [13] considered another Hessian typed equation related to the Gauduchon conjecture

$$(1.4) \quad \det(\omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2}) = e^F \det(\omega^{n-1})$$

$$\omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2} > 0, \sup_M u = 0,$$

where ω_0 and ω are two Hermitian metrics on M .

In [13], Tosatti-Weinkove solved equation (1.4) if ω is Kähler. One of the main parts is doing the second order estimate. They use the similar auxiliary function in [4]. Later on, in [14], they can solve (1.4) if ω is Hermitian. The second order estimate becomes more difficult in the Hermitian case, the authors succeeded to obtain the second order estimates by modifying the auxiliary function in [4].

In this paper, we solve equation (1.1) on closed Hermitian manifolds. More precisely, our main result is

Theorem 1.1. *Let (M, g) be a closed Hermitian manifold of complex dimension $n \geq 2$, f is a smooth real function on M . Then there is a unique real number b and a unique smooth real function u on M solving*

$$(1.5) \quad C_n^k \omega_u^k \wedge \omega^{n-k} = e^{f+b} \omega^n$$

$$\omega_u \in \Gamma_k(M), \sup_M u = 0.$$

We use the continuity method to solve the problem (1.5). The openness follows from implicit function theory. The closeness argument can be reduced to *a priori* estimates up to the second by the standard Evans-Krylov theory. Actually, we can derive the zero order estimate and the second order estimate of solutions of equation (1.1) and thus use a blow up method to obtain the gradient estimate.

In [11], Tosatti-Weinkove derived the key zero order estimate by proving a Cherrier-type inequality which was originally proved in [1]. For equation (1.1), we can prove the similar Cherrier-type inequality but the analysis becomes a bit complicated since ω_u may not be positive. Some inequalities for k -th elementary symmetric functions in [2] are needed. For the second order estimate, the main difficulty is that there are new terms of the form $T * D^3u$, where T is the torsion of ω and D^3u represents the third derivatives of u . To control these terms, we use the auxiliary function due to Tosatti-Weinkove in [14]. The main difference is that for equation (1.1) we need to use some lemmas for k -th elementary symmetric functions proved by Hou-Ma-Wu in [4].

The rest of the paper is organized as follows. In section 2, we give some preliminaries. In section 3, the Cherrier-type inequality is derived, thus we obtain the C^0 estimate. In section 4, we will prove the second order estimate by a similar auxiliary function in [14].

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2. PRELIMINARIES

Let (M, g) be a compact Hermitian manifold and let ∇ denote the Chern connection of g . In this section we will give some preliminaries about the k -th elementary symmetric function and the commutation formula of covariant derivatives.

2.1. Elementary symmetric function. The k -th elementary symmetric function is defined by

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

where $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. Let $\lambda(a_{i\bar{j}})$ denote the eigenvalues of Hermitian matrix $\{a_{i\bar{j}}\}$, we define

$$\sigma_k(a_{i\bar{j}}) = \sigma_k(\lambda(a_{i\bar{j}})).$$

The definition of σ_k can be naturally extended to Hermitian manifold. Indeed, let $A^{1,1}(M, \mathbb{R})$ be the space of smooth real $(1, 1)$ -forms on M , for $\chi \in A^{1,1}(M, \mathbb{R})$ we define

$$\sigma_k(\chi) = \left(\frac{n}{k}\right) \frac{\chi^k \wedge \omega^{n-k}}{\omega^n}.$$

Definition 2.1.

$$(2.1) \quad \Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \dots, k\}.$$

Similarly, we define Γ_k on M as follows

$$(2.2) \quad \Gamma_k(M) := \{\chi \in A^{1,1}(M, \mathbb{R}^n) : \sigma_j(\chi) > 0, j = 1, \dots, k\}.$$

Furthermore, $\sigma_r(\lambda|i_1 \dots i_l)$, with i_1, \dots, i_l being distinct, stands for the r -th symmetric function with $\lambda_{i_1} = \dots = \lambda_{i_l} = 0$. For more details about elementary symmetric functions, one can see the lecture notes [15].

To prove the C^0 estimate, we need the following lemma of elementary symmetric functions.

Lemma 2.2. *Suppose that $\lambda \in \Gamma_k$, $3 \leq k \leq n$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, then there exists a positive constant C depending only on k and n , such that for $0 \leq i \leq k-2$.*

$$(2.3) \quad \begin{aligned} |\lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_i}| &\leq C \sigma_i(\lambda|j), \\ 1 \leq j_1 < j_2 < \dots < j_i \leq n, j_l &\neq j, 1 \leq l \leq i, 1 \leq j \leq n. \end{aligned}$$

Proof. Since

$$\sum_{p=k}^n \lambda_p = \sigma_1(\lambda|12 \cdots k-1) > 0,$$

and

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

then

$$(2.4) \quad |\lambda_p| \leq (n-k) \lambda_k, k+1 \leq p \leq n.$$

We first prove the lemma for $k = 3$. In this case, it needs to prove that there exists a constant C such that

$$|\lambda_l| \leq C \sigma_1(\lambda|j),$$

for $1 \leq j, l \leq n$ and $l \neq j$. Indeed, $\sigma_1(\lambda|j) = \lambda_l + \sigma_1(\lambda|jl)$, thus $\lambda_l \leq \sigma_1(\lambda|j)$. Now, we assume $\lambda_l < 0$, then $l \geq 4$. By (2.4), we have

$$|\lambda_l| \leq (n-3) \lambda_3, 4 \leq l \leq n.$$

Since $\lambda|j \in \Gamma_2$, by the proof in [2] which used the result in [7], there exists a constant θ_1 such that $\sigma_1(\lambda|j) \geq \theta_1 \lambda_2$ if $j = 1$ and $\sigma_1(\lambda|j) \geq \theta_1 \lambda_1$ if $2 \leq j \leq n$. Taking $C = \frac{n-3}{\theta_1}$, we then prove the lemma for the case $k = 3$.

Next we prove the lemma for the general $k, 3 \leq k \leq n$.

If $j > i$, by the result in [15]

$$\sigma_i(\lambda|j) \geq \theta(n, k) \lambda_1 \cdots \lambda_i.$$

Thus we have

$$\begin{aligned} |\lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_i}| &= \lambda_{j_1} \cdots \lambda_{j_q} |\lambda_{j_{q+1}} \cdots \lambda_{j_i}| \leq \lambda_1 \cdots \lambda_q (n-k)^{i-q} \lambda_k^{i-q} \\ &\leq (n-k)^i \lambda_1 \cdots \lambda_i \leq \frac{(n-k)^i}{\theta(n, k)} \sigma_i(\lambda|j). \end{aligned}$$

If $j \leq i$, then similarly

$$\sigma_i(\lambda|j) \geq \theta(n, k) \lambda_1 \cdots \lambda_{j-1} \lambda_{j+1} \cdots \lambda_{i+1}.$$

Thus we have

$$\begin{aligned} |\lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_i}| &= \lambda_{j_1} \cdots \lambda_{j_q} |\lambda_{j_{q+1}} \cdots \lambda_{j_i}| \leq \lambda_1 \cdots \lambda_{j-1} \lambda_{j+1} \cdots \lambda_{q+1} (n-k)^{i-q} \lambda_k^{i-q} \\ &\leq (n-k)^i \lambda_1 \cdots \lambda_{j-1} \lambda_{j+1} \cdots \lambda_{i+1} \leq \frac{(n-k)^i}{\theta(n, k)} \sigma_i(\lambda|j). \end{aligned}$$

□

Using this lemma, we immediately obtain the following lemma which is a key ingredient for proving lemma 3.2.

Lemma 2.3.

$$(2.5) \quad \sum_{i=0}^{k-2} \left| \frac{\sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega_u^i \wedge T_i}{\omega^n} \right| \leq C \sum_{i=0}^{k-2} \frac{\sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega_u^i \wedge \omega^{n-i-1}}{\omega^n},$$

,where T_i is defined as the combinations of $\omega, \partial\omega, \partial\bar{\partial}\omega$, more precisely

$$T_i = \sum_{0 \leq 3p+2q \leq n-i} \omega^{n-i-3p-2q} \wedge (\sqrt{-1})^p (\partial\omega)^p \wedge (\bar{\partial}\omega)^q \wedge (\sqrt{-1})^q (\partial\bar{\partial}\omega)^q$$

Proof. For $x \in M$, we choose the coordinates such that

$$\omega(x) = \sum_{j=1}^n dz^j \wedge d\bar{z}^j, \omega_u(x) = \sum_{j=1}^n \lambda_j dz^j \wedge d\bar{z}^j,$$

and

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Thus we have

$$\begin{aligned} (2.6) \quad \sum_{i=0}^{k-2} \left| \frac{\sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega_u^i \wedge T_i}{\omega^n} \right| &\leq C \sum_{i=0}^{k-2} \sum_{1 \leq j_1 < \cdots < j_i \leq n, \neq j, l} |u_j| |u_l| |\lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_i}| \\ &\leq C \sum_{i=0}^{k-2} \sum_{j=1}^n \sum_{\substack{1 \leq j_1 < \cdots < j_i \leq n \\ j_l \neq j}} |u_j|^2 |\lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_i}| \\ &\leq C \sum_{i=0}^{k-2} \sum_{j=1}^n \sigma_i(\lambda | j) |u_j|^2 \\ &= C \sum_{i=0}^{k-2} \frac{\sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega_u^i \wedge \omega^{n-i-1}}{\omega^n}, \end{aligned}$$

where we have used the lemma 2.1 in the last inequality. \square

2.2. Commutation formula of covariant derivatives. In local complex coordinates z_1, \cdots, z_n , we have

$$(2.7) \quad g_{i\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right), \{g^{i\bar{j}}\} = \{g_{i\bar{j}}\}^{-1}$$

For the Chern connection ∇ , we denote the covariant derivatives as follows:

$$(2.8) \quad u_i = \nabla_{\frac{\partial}{\partial z^i}} u, u_{i\bar{j}} = \nabla_{\frac{\partial}{\partial \bar{z}^j}} \nabla_{\frac{\partial}{\partial z^i}} u, u_{i\bar{j}k} = \nabla_{\frac{\partial}{\partial z^k}} \nabla_{\frac{\partial}{\partial \bar{z}^j}} \nabla_{\frac{\partial}{\partial z^i}} u$$

we use the following commutation formula for covariant derivatives on Hermitian manifolds which can be founded in [14]:

$$\begin{aligned} (2.9) \quad u_{i\bar{j}l} &= u_{l\bar{j}i} - T_{li}^p u_{p\bar{j}} \\ u_{p\bar{i}j} &= u_{p\bar{j}i} + u_q R_{i\bar{j}p}^q \\ u_{i\bar{p}j} &= u_{i\bar{j}p} - \bar{T}_{jp}^q u_{i\bar{q}}. \end{aligned}$$

$$(2.10) \quad u_{i\bar{j}l\bar{m}} = u_{l\bar{m}i\bar{j}} + u_{p\bar{j}} R_{l\bar{m}i}^p - u_{p\bar{m}} R_{i\bar{j}l}^p - T_{li}^p u_{p\bar{m}j} - \bar{T}_{mj}^p u_{l\bar{p}i} - T_{li}^p \bar{T}_{mj}^q u_{p\bar{q}}$$

For the details we recommend the reader to the reference [14].

3. ZERO ORDER ESTIMATE

In this section we derive the zero order estimate by proving a Cherrier-type inequality and the lemmas in [11]. Since the constant b in Theorem 1.1 satisfies

$$|b| \leq \sup |f| + C,$$

where C is a positive constant depending only on (M, ω) . Thus, we will assume $b = 0$ for convenience.

Theorem 3.1. *Let u be a solution of Theorem 1.1. Then there exists a constant C depending only on (M, ω) and $\sup_M |f|$ such that*

$$\sup_M |u| \leq C.$$

Due to Tosatti-Weinkove's results, the zero order estimate can be reduced to derive a Cherrier-type inequality which was firstly proved in Cherrier's paper [1]. For the Hessian equation, the analysis becomes a bit complicated in the lack of the positivity of ω_u . Recently¹, Sun [8] also proved the following lemma for $k = 2$ and $k \geq 3$ under some extra conditions.

Lemma 3.2. *There exist constants p_0 and C depending only on (M, ω) such that for any $p \geq p_0$*

$$\int_M |de^{-\frac{p}{2}u}|_g^2 \omega^n \leq Cp \int_M e^{-pu} \omega^n$$

Proof. By the equation, we have

$$\omega_u^k \wedge \omega^{n-k} - \omega^n = (e^f - 1)\omega^n \leq C_0 \omega^n,$$

where C_0 is a constant depending only on f .

On the other hand,

$$(3.1) \quad \omega_u^k \wedge \omega^{n-k} - \omega^n = (\omega_u^k - \omega^k) \wedge \omega^{n-k} = \sqrt{-1} \partial \bar{\partial} u \wedge \alpha,$$

where $\alpha = \sum_{i=1}^k \omega_u^{i-1} \wedge \omega^{n-i}$.

¹The author independently proved the C^0 estimate before [8] was posted on arXiv.

Now multiply both sides in (3.1) by e^{-pu} and integrate by parts ,

$$\begin{aligned}
 (3.2) \quad C_0 \int_M e^{-pu} \omega^n &\geq \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \alpha \\
 &= - \int_M \partial e^{-pu} \sqrt{-1} \bar{\partial} u \wedge \alpha + \int_M e^{-pu} \sqrt{-1} \bar{\partial} u \wedge \partial \alpha \\
 &= p \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \alpha - \frac{1}{p} \int_M \sqrt{-1} \bar{\partial} e^{-pu} \wedge \partial \alpha \\
 &= p \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \alpha + \frac{1}{p} \int_M e^{-pu} \sqrt{-1} \bar{\partial} \partial \alpha \\
 &:= A + B,
 \end{aligned}$$

where we denote

$$\begin{aligned}
 A &= p \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \left(\sum_{i=1}^k \omega_u^{i-1} \wedge \omega^{n-i} \right) \\
 B &= \frac{1}{p} \int_M e^{-pu} \sqrt{-1} \bar{\partial} \partial \alpha.
 \end{aligned}$$

We will use the term A to control the terms B . Direct calculation gives

$$\partial \alpha = n \sum_{i=1}^{k-1} \omega_u^{i-1} \wedge \omega^{n-i-1} \wedge \partial \omega + (n-k) \omega_u^{k-1} \wedge \omega^{n-k-1} \wedge \partial \omega$$

$$\begin{aligned}
 \bar{\partial} \partial \alpha &= (n-k)(n-k-1) \omega_u^{k-1} \wedge \omega^{n-k-2} \wedge \bar{\partial} \omega \wedge \partial \omega + (n-k) \omega_u^{k-1} \wedge \omega^{n-k-1} \wedge \bar{\partial} \partial \omega \\
 &\quad + (n-k)(n+k-1) \omega_u^{k-2} \wedge \omega^{n-k-1} \wedge \bar{\partial} \omega \wedge \partial \omega \\
 &\quad + n(n-1) \sum_{i=0}^{k-3} \omega_u^i \wedge \omega^{n-i-3} \wedge \bar{\partial} \omega \wedge \partial \omega + n \sum_{i=1}^{k-2} \omega_u^i \wedge \omega^{n-i-2} \wedge \bar{\partial} \partial \omega
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 B &= \frac{(n-k)(n-k-1)}{p} \int_M \sqrt{-1} e^{-pu} \omega_u^{k-1} \wedge \omega^{n-k-2} \wedge \bar{\partial} \omega \wedge \partial \omega \\
 &\quad + \frac{(n-k)}{p} \int_M \sqrt{-1} e^{-pu} \omega_u^{k-1} \wedge \omega^{n-k-1} \wedge \bar{\partial} \partial \omega \\
 &\quad + \frac{(n+k-1)(n-k)}{p} \int_M \sqrt{-1} e^{-pu} \omega_u^{k-2} \wedge \omega^{n-k-1} \wedge \bar{\partial} \omega \wedge \partial \omega \\
 &\quad + \frac{n(n-1)}{p} \sum_{i=0}^{k-3} \int_M \sqrt{-1} e^{-pu} \omega_u^i \wedge \omega^{n-i-3} \wedge \bar{\partial} \omega \wedge \partial \omega + \frac{n}{p} \sum_{i=1}^{k-2} \int_M \sqrt{-1} e^{-pu} \omega_u^i \wedge \omega^{n-i-2} \wedge \bar{\partial} \partial \omega
 \end{aligned}$$

When $k = 2$, the term B just becomes

$$\begin{aligned}
(3.3) \quad B &= \frac{(n-2)(n-3)}{p} \int_M \sqrt{-1} e^{-pu} \omega_u \wedge \omega^{n-4} \wedge \bar{\partial} \omega \wedge \partial \omega + \frac{(n-2)}{p} \int_M \sqrt{-1} e^{-pu} \omega_u \wedge \omega^{n-3} \wedge \bar{\partial} \partial \omega \\
&\quad + \frac{(n+1)(n-2)}{p} \int_M \sqrt{-1} e^{-pu} \omega^{n-3} \wedge \bar{\partial} \omega \wedge \partial \omega \\
&= \frac{(n-2)(n-3)}{p} \int_M \sqrt{-1} e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-4} \wedge \bar{\partial} \omega \wedge \partial \omega + \frac{(n-2)}{p} \int_M \sqrt{-1} e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-3} \wedge \bar{\partial} \partial \omega \\
&\quad + \frac{2(n-1)(n-2)}{p} \int_M \sqrt{-1} e^{-pu} \omega^{n-3} \wedge \bar{\partial} \omega \wedge \partial \omega + \frac{(n-2)}{p} \int_M \sqrt{-1} e^{-pu} \omega^{n-2} \wedge \bar{\partial} \partial \omega \\
&\geq \frac{(n-2)(n-3)}{p} \int_M \sqrt{-1} e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-4} \wedge \bar{\partial} \omega \wedge \partial \omega \\
&\quad + \frac{(n-2)}{p} \int_M \sqrt{-1} e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-3} \wedge \bar{\partial} \partial \omega - \frac{C_1}{p} \int_M e^{-pu} \omega^n
\end{aligned}$$

We next use integration by parts again to deal with the first term and second term on the right hand side of the above equality. Indeed,

$$\begin{aligned}
(3.4) \quad &\int_M \sqrt{-1} e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-4} \wedge \bar{\partial} \omega \wedge \partial \omega \\
&= p \int_M \sqrt{-1} e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega^{n-4} \wedge \bar{\partial} \omega \wedge \partial \omega + \int_M \sqrt{-1} e^{-pu} \sqrt{-1} \bar{\partial} u \wedge \partial(\omega^{n-4} \wedge \bar{\partial} \omega \wedge \partial \omega) \\
&= p \int_M \sqrt{-1} e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega^{n-4} \wedge \bar{\partial} \omega \wedge \partial \omega + \frac{1}{p} \int_M \sqrt{-1} e^{-pu} \sqrt{-1} \bar{\partial} \partial(\omega^{n-4} \wedge \bar{\partial} \omega \wedge \partial \omega) \\
&\geq -pC_1 \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega^{n-1} - \frac{C_1}{p} \int_M e^{-pu} \omega^n \\
&\geq -C_1 A - \frac{C_1}{p} \int_M e^{-pu} \omega^n
\end{aligned}$$

The similar calculation gives

$$(3.5) \quad \int_M \sqrt{-1} e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-3} \wedge \bar{\partial} \partial \omega \geq -C_1 A - \frac{C_1}{p} \int_M e^{-pu} \omega^n$$

Inserting (3.4) and (3.5) into (3.3), we have

$$B \geq -\frac{C_1}{p} A - \frac{C_1}{p} \int_M e^{-pu} \omega^n$$

By (3.2) and choosing $p_0 = 2C_1 + 1$, we obtain for $p \geq p_0$

$$\frac{A}{2} \leq (1 - \frac{C_1}{p})A \leq (\frac{C_1}{p} + C_0) \int_M e^{-pu} \omega^n \leq (C_0 + 1) \int_M e^{-pu} \omega^n$$

By (3.7) in the next page, we thus prove the lemma.

For the general $k, 3 \leq k \leq n$, we claim that there exist constants C_{1i} depending only on $n, k, (M, \omega)$ such that the following holds for $0 \leq i \leq k-1$,

$$(3.6) \quad \int_M e^{-pu} \omega_u^i \wedge T_i \geq -pC_{1i} \sum_{j=0}^{k-2} \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega_u^j \wedge \omega^{n-j-1} - C_{1i} \int_M e^{-pu} \omega^n$$

, where T_i is defined as the combinations of $\omega, \partial\omega, \partial\bar{\partial}\omega$, more precisely

$$T_i = \sum_{0 \leq 3p+2q \leq n-i} \omega^{n-i-3p-2q} \wedge (\sqrt{-1})^p (\partial\omega)^p \wedge (\bar{\partial}\omega)^p \wedge (\sqrt{-1})^q (\partial\bar{\partial}\omega)^q$$

We use the claim (3.6) to prove the lemma

$$\begin{aligned} B &\geq -C_1 \sum_{i=2}^k \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega_u^{k-i} \wedge \omega^{n+i-k-1} - \frac{C_1}{p} \int_M e^{-pu} \omega^n \\ &\geq -\frac{C_1}{p} A - \frac{C_1}{p} \int_M e^{-pu} \omega^n \end{aligned}$$

Thus we have

$$(1 - \frac{C_1}{p})A \leq (\frac{C_1}{p} + C_0) \int_M e^{-pu} \omega^n$$

Now we choose $p_0 = 2C_1 + 1$, then for any $p \geq p_0$,

$$p^2 \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega^{n-1} \leq 2p(C_0 + 1) \int_M e^{-pu} \omega^n$$

Therefore we have

$$\begin{aligned} (3.7) \quad \int_M |\partial e^{-\frac{p}{2}u}|_g^2 \omega^n &= \frac{np^2}{4} \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega^{n-1} \\ &\leq \frac{np(C_0 + 1)}{2} \int_M e^{-pu} \omega^n = pC \int_M e^{-pu} \omega^n \end{aligned}$$

Now, we prove the claim (3.6) by inductive argument.

When $i = 1$, we have

$$\begin{aligned}
\int_{\mathbf{M}} e^{-pu} \omega_u \wedge T_1 &= \int_{\mathbf{M}} e^{-pu} \omega \wedge T_1 + \int_{\mathbf{M}} e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge T_1 \\
&= \int_{\mathbf{M}} e^{-pu} \omega \wedge T_1 - \int_{\mathbf{M}} \partial e^{-pu} \wedge \sqrt{-1} \bar{\partial} u \wedge T_1 + \int_{\mathbf{M}} e^{-pu} \sqrt{-1} \bar{\partial} u \wedge \partial T_1 \\
&= \int_{\mathbf{M}} e^{-pu} \omega \wedge T_1 + p \int_{\mathbf{M}} e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge T_1 - \frac{1}{p} \int_{\mathbf{M}} \sqrt{-1} \bar{\partial} e^{-pu} \wedge \partial T_1 \\
&= p \int_{\mathbf{M}} e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge T_1 + \int_{\mathbf{M}} e^{-pu} \omega \wedge T_1 - \frac{1}{p} \int_{\mathbf{M}} e^{-pu} \wedge \sqrt{-1} \partial \bar{\partial} T_1 \\
&\geq -C_1 p \int_{\mathbf{M}} e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge T_1 - C_1 \int_{\mathbf{M}} e^{-pu} \omega^n
\end{aligned}$$

Suppose that the claim is true for $l \leq i-1$, we will prove that the claim is also true for $l = i$. Indeed,

$$\begin{aligned}
\int_{\mathbf{M}} e^{-pu} \omega_u^i \wedge T_i &= \int_{\mathbf{M}} e^{-pu} \omega_u^{i-1} \wedge \omega \wedge T_i + \int_{\mathbf{M}} e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \omega_u^{i-1} \wedge T_i \\
&= \int_{\mathbf{M}} e^{-pu} \omega_u^{i-1} \wedge \omega \wedge T_i + p \int_{\mathbf{M}} e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega_u^{i-1} \wedge T_i \\
&\quad + \int_{\mathbf{M}} e^{-pu} \bar{\partial} u \wedge \sqrt{-1} \partial (\omega_u^{i-1} \wedge T_i) \\
&:= A_{i,1} + A_{i,2} + A_{i,3}
\end{aligned}$$

By the induction ,

$$\begin{aligned}
A_{i,1} &= \int_{\mathbf{M}} e^{-pu} \omega_u^{i-1} \wedge \omega \wedge T_i \\
&\geq -p C_{1i}(n, k, \omega) \sum_{j=0}^{k-2} \int_{\mathbf{M}} e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega_u^j \wedge \omega^{n-j-1} - C_{1i}(n, k, \omega) \int_{\mathbf{M}} e^{-pu} \omega^n
\end{aligned}$$

By the inequality (2.5) in lemma 2.3, we have

(3.8)

$$A_{i,2} = p \int_{\mathbf{M}} e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega_u^{i-1} \wedge T_i \geq -p C_{2i} \int_{\mathbf{M}} e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega_u^{i-1} \wedge \omega^{n-i}$$

Now we deal with the term $A_{i,3}$,

$$\begin{aligned}
A_{i,3} &= \int_M e^{-pu} \bar{\partial} u \wedge \sqrt{-1} \partial (\omega_u^{i-1} \wedge T_i) = \frac{1}{p} \int_M e^{-pu} \sqrt{-1} \bar{\partial} \partial (\omega_u^{i-1} \wedge T_i) \\
&= \frac{(i-1)(i-2)}{p} \int_M e^{-pu} \sqrt{-1} \omega_u^{i-3} \wedge \bar{\partial} \omega \wedge \partial \omega \wedge T_i + \frac{i-1}{p} \int_M e^{-pu} \omega_u^{i-2} \wedge \sqrt{-1} \bar{\partial} (\partial \omega \wedge T_i) \\
&\quad + \frac{i-1}{p} \int_M e^{-pu} \omega_u^{i-2} \wedge \sqrt{-1} \bar{\partial} \omega \wedge \partial T_i - \frac{1}{p} \int_M e^{-pu} \omega_u^{i-1} \wedge \sqrt{-1} \bar{\partial} \partial T_i \\
&= \frac{(i-1)(i-2)}{p} \int_M e^{-pu} \sqrt{-1} \omega_u^{i-3} \wedge \bar{\partial} \omega \wedge \partial \omega \wedge T_i \\
&\quad + \frac{i-1}{p} \int_M e^{-pu} \omega_u^{i-2} \wedge [\sqrt{-1} \bar{\partial} (\partial \omega \wedge T_i) + \sqrt{-1} \bar{\partial} \omega \wedge \partial T_i] - \frac{1}{p} \int_M e^{-pu} \omega_u^{i-1} \wedge \sqrt{-1} \bar{\partial} \partial T_i \\
&\geq -pC_{3i} \sum_{j=0}^{k-2} \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega_u^j \wedge \omega^{n-j-1} - C_{3i}(n, k, \omega) \int_M e^{-pu} \omega^n.
\end{aligned}$$

For the last inequality, we have used the induction. \square

4. SECOND ORDER ESTIMATE

In this section we use the auxiliary function in [14] which is modified by the auxiliary function in [4] to derive the second order estimate of the form (1.3). The difficult part arises from the third order derivatives' Locally the equation is

$$(4.1) \quad \sigma_k(\omega_u) = e^f.$$

Theorem 4.1. *There exists a uniform constant C depending only on (M, ω) and f such that*

$$(4.2) \quad \max |\partial \bar{\partial} u|_g \leq C(1 + \max |\nabla u|_g^2)$$

Proof. Denote $w_{i\bar{j}} = g_{i\bar{j}} + u_{i\bar{j}}$ and let $\xi \in T^{1,0}M$, $|\xi|_g^2 = 1$.

We use the auxiliary function which is similar to the one in [14]

$$H(x, \xi) = \log(w_{k\bar{l}} \xi^k \bar{\xi}^l) + c_0 \log(g^{k\bar{l}} w_{p\bar{l}} w_{k\bar{q}} \xi^p \bar{\xi}^q) + \varphi(|\nabla u|_g^2) + \psi(u),$$

where φ, ψ are given by

$$\begin{aligned}
\varphi(s) &= -\frac{1}{2} \log \left(1 - \frac{s}{2K} \right), \quad 0 \leq s \leq K-1, \\
\psi(t) &= -A \log \left(1 + \frac{t}{2L} \right), \quad -L+1 \leq t \leq 0,
\end{aligned}$$

for

$$K := \sup_M |\nabla u|_g^2 + 1, \quad L = \sup_M |u| + 1, \quad A := 2L(C_0 + 1),$$

where A_0 is a constant to be determined later. c_0 is a small positive constant depending only on n and will be determined later. By [4], we have

$$(4.3) \quad \frac{1}{2K} \geq \varphi' \geq \frac{1}{4K} > 0, \quad \varphi'' = 2(\varphi')^2 > 0.$$

$$(4.4) \quad \frac{A}{L} \geq -\psi' \geq \frac{A}{2L} = C_0 + 1, \quad \psi'' \geq \frac{2\varepsilon_0}{1-\varepsilon_0}(\psi')^2, \quad \text{for all } \varepsilon_0 \leq \frac{1}{2A+1}.$$

These inequalities will be used below.

Suppose $H(x, \xi)$ attains its maximum at the point x_0 in the direction ξ_0 , then we choose local coordinates $\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}\}$ near x_0 such that

$$g_{i\bar{j}}(x_0) = \delta_{ij}, \quad u_{i\bar{j}} = u_{i\bar{i}}(x_0)\delta_{ij}, \\ \lambda_i = w_{i\bar{i}}(x_0) = 1 + u_{i\bar{i}}(x_0) \text{ with } \lambda_1 \geq \dots \geq \lambda_n.$$

We will prove that

$$H(x_0, \xi) \leq H(x_0, \frac{\partial}{\partial z^1}) \quad \forall \xi \in T^{1,0}M, \quad |\xi|_g^2 = 1, \quad \sum_{i,j} w_{i\bar{j}}(x_0) \xi^i \bar{\xi}^j > 0$$

by choosing c_0 small enough. In fact, at x_0 we have

$$\log(w_{k\bar{l}} \xi^k \bar{\xi}^l) + c_0 \log(g^{k\bar{l}} w_{p\bar{l}} w_{k\bar{q}} \xi^p \bar{\xi}^q) = \log\left(\sum_{k=1}^n w_{k\bar{k}} |\xi^k|^2\right) + c_0 \log\left(\sum_{k=1}^n |w_{k\bar{k}}|^2 |\xi^k|^2\right)$$

If $w_{n\bar{n}} \geq -w_{1\bar{1}}$ which is always satisfied when $n \leq 3$, we have $w_{i\bar{i}}^2 \leq w_{1\bar{1}}$. Thus we have $H(x_0, \xi) \leq H(x_0, \frac{\partial}{\partial z^1})$.

Now we suppose that $w_{n\bar{n}} < -w_{1\bar{1}}$, thus we have $n \geq 4$. Let i_0 be the smallest integer satisfying $w_{i\bar{i}} < -w_{1\bar{1}}$, then $i_0 \geq k+1$. By $|w_{i\bar{i}}| < (n-2)w_{1\bar{1}}$ we have

$$\begin{aligned} & \log\left(\sum_{i=1}^n w_{i\bar{i}} |\xi^i|^2\right) + c_0 \log\left(\sum_{i=1}^n |w_{i\bar{i}}|^2 |\xi^i|^2\right) \\ & \leq \log w_{1\bar{1}} \left(\sum_{i=1}^{i_0-1} |\xi^i|^2 - \sum_{i=i_0}^n |\xi^i|^2\right) + c_0 \log(w_{1\bar{1}}^2 \sum_{i=1}^{i_0-1} |\xi^i|^2 + (n-2)^2 w_{1\bar{1}}^2 \sum_{i=i_0}^{i_0-1} |\xi^i|^2) \\ & = \log w_{1\bar{1}} (1-2t) + c_0 \log w_{1\bar{1}}^2 (1-t + (n-2)^2 t) := h(t), \end{aligned}$$

where $t = \sum_{i=i_0}^n |\xi^i|^2 \in (0, \frac{1}{2})$.

By choosing $c_0 = \frac{2}{(n-2)^2-1}$, we have $h'(t) \leq 0$, thus

$$h(t) \leq h(0) = \log(w_{1\bar{1}}) + c_0 \log w_{1\bar{1}}^2.$$

Consequently, we have proved

$$H(x_0, \xi) \leq H(x_0, \frac{\partial}{\partial z^1}), \quad \text{for } \forall \xi \in T^{1,0}M, \quad |\xi|_g^2 = 1, \quad \sum_{i,j} \eta_{i\bar{j}}(x_0) \xi^i \bar{\xi}^j > 0,$$

by choosing $c_0 = \frac{2}{(n-2)^2-1}$ when $n \geq 4$ and $c_0 = 1$ when $n \leq 3$.

We extend ξ_0 near x_0 as

$$\xi_0 = (g_{1\bar{1}})^{\frac{1}{2}} \frac{\partial}{\partial z^1}.$$

Consider the function

$$Q(x) = H(x, \xi_0) = \log(g_{1\bar{1}}^{-1} w_{1\bar{1}}) + c_0 \log(g_{1\bar{1}}^{-1} g^{k\bar{l}} w_{1\bar{l}} w_{k\bar{1}}) + \varphi(|\nabla u|_g^2) + \psi(u).$$

We will calculate $F^{i\bar{j}} Q_{i\bar{j}}$ at x_0 to get the estimate, all the calculations are taken at x_0 . For simplicity, we denote $\xi = \xi_0$ in the following. By $\langle \xi, \bar{\xi} \rangle_g = |\xi|_g^2 = 1$, differentiating both sides, we obtain at x_0

$$\begin{aligned} 0 &= \frac{\partial}{\partial z^i} \langle \xi, \bar{\xi} \rangle_g = \langle \nabla_{\frac{\partial}{\partial z^i}} \xi, \bar{\xi} \rangle_g + \langle \xi, \nabla_{\frac{\partial}{\partial z^i}} \bar{\xi} \rangle_g \\ &= \langle \xi^k_{,i} \frac{\partial}{\partial z^k}, \bar{\xi}^l \frac{\partial}{\partial z^l} \rangle_g + \langle \xi^k \frac{\partial}{\partial z^k}, \bar{\xi}^l_{,i} \frac{\partial}{\partial z^l} \rangle_g \\ &= g_{k\bar{l}} \xi^k_{,i} \bar{\xi}^l + g_{k\bar{l}} \xi^k \bar{\xi}^l_{,i} \\ (4.5) \quad &= \xi^1_{,i} + \bar{\xi}^1_{,i}. \end{aligned}$$

We also have the basic formula for $\xi \in T^{1,0}M$:

$$\begin{aligned} \bar{\xi}^k_{,i} &= \frac{\partial \bar{\xi}^k}{\partial z^i} = \overline{\frac{\partial \xi^k}{\partial z^i}} = \overline{\xi^k_{,i}}, \\ \xi^k_{,\bar{i}} &= \frac{\partial \xi^k}{\partial \bar{z}^i} = \overline{\frac{\partial \bar{\xi}^k}{\partial \bar{z}^i}} = \overline{\bar{\xi}^k_{,\bar{i}}}, \\ \bar{\xi}^k_{,i} &= \frac{\partial \bar{\xi}^k}{\partial z^i} = \overline{\frac{\partial \xi^k}{\partial z^i}} = \overline{\xi^k_{,i}}, \\ (4.6) \quad \xi^k_{,\bar{i}} &= \frac{\partial \xi^k}{\partial \bar{z}^i} = \overline{\frac{\partial \bar{\xi}^k}{\partial \bar{z}^i}} = \overline{\bar{\xi}^k_{,\bar{i}}} \end{aligned}$$

Direct calculations give

$$\begin{aligned} Q_i &= \frac{(w_{k\bar{l}} \xi^k \bar{\xi}^l)_i}{w_{k\bar{l}} \xi^k \bar{\xi}^l} + c_0 \frac{(g^{p\bar{q}} w_{k\bar{q}} w_{p\bar{l}} \xi^k \bar{\xi}^l)_i}{g^{p\bar{q}} w_{k\bar{q}} w_{p\bar{l}} \xi^k \bar{\xi}^l} + \varphi_i + \psi_i \\ Q_{i\bar{i}} &= \frac{(w_{k\bar{l}} \xi^k \bar{\xi}^l)_{i\bar{i}}}{w_{k\bar{l}} \xi^k \bar{\xi}^l} - \frac{(w_{k\bar{l}} \xi^k \bar{\xi}^l)_i (w_{k\bar{l}} \xi^k \bar{\xi}^l)_{\bar{i}}}{(w_{k\bar{l}} \xi^k \bar{\xi}^l)^2} + c_0 \frac{(g^{p\bar{q}} w_{k\bar{q}} w_{p\bar{l}} \xi^k \bar{\xi}^l)_{i\bar{i}}}{g^{p\bar{q}} w_{k\bar{q}} w_{p\bar{l}} \xi^k \bar{\xi}^l} \\ &\quad - c_0 \frac{(g^{p\bar{q}} w_{k\bar{q}} w_{p\bar{l}} \xi^k \bar{\xi}^l)_i (g^{p\bar{q}} w_{k\bar{q}} w_{p\bar{l}} \xi^k \bar{\xi}^l)_{\bar{i}}}{(g^{p\bar{q}} w_{k\bar{q}} w_{p\bar{l}} \xi^k \bar{\xi}^l)^2} + \varphi_{i\bar{i}} + \psi_{i\bar{i}} \end{aligned}$$

Next, we will simplify Q_i and $Q_{\bar{i}}$.

By (4.5), we have

$$\begin{aligned} \left(w_{k\bar{l}} \xi^k \bar{\xi}^l \right)_i &= w_{k\bar{l},i} \xi^k \bar{\xi}^l + w_{k\bar{l}} \xi^k_{,i} \bar{\xi}^l + w_{k\bar{l}} \xi^k \bar{\xi}^l_{,i} \\ &= w_{1\bar{1},i} + w_{1\bar{1}} \left(\xi^1_{,i} + \bar{\xi}^1_{,i} \right) \\ &= w_{1\bar{1}i}, \end{aligned}$$

Thus we have

$$\begin{aligned} \left(g^{p\bar{q}} w_{k\bar{q}} w_{p\bar{l}} \xi^k \bar{\xi}^l \right)_i &= g^{p\bar{q}} w_{k\bar{q}} w_{p\bar{l}} \xi^k \bar{\xi}^l + g^{p\bar{q}} w_{k\bar{q}} w_{p\bar{l}} \xi^k_{,i} \bar{\xi}^l + g^{p\bar{q}} w_{k\bar{q}} w_{p\bar{l}} \xi^k \bar{\xi}^l_{,i} + g^{p\bar{q}} w_{k\bar{q}} w_{p\bar{l}} \xi^k_{,i} \bar{\xi}^l_{,i} \\ &= w_{1\bar{1}} (w_{1\bar{1}i} + w_{1\bar{1}i}) + w_{1\bar{1}}^2 \left(\xi^1_{,i} + \bar{\xi}^1_{,i} \right) \\ &= 2w_{1\bar{1}} w_{1\bar{1}i}. \end{aligned}$$

Therefore, we obtain the simplified formula for the term Q_i at x_0 .

$$(4.7) \quad Q_i = \frac{w_{1\bar{1}i}}{w_{1\bar{1}}} + c_0 \frac{2w_{1\bar{1}i}}{w_{1\bar{1}}} + \varphi_i + \psi_i = (1 + 2c_0) \frac{w_{1\bar{1}i}}{w_{1\bar{1}}} + \varphi_i + \psi_i = 0$$

Similar calculations give

$$\begin{aligned} \left(w_{k\bar{l}} \xi^k \bar{\xi}^l \right)_{\bar{i}\bar{i}} &= \left[w_{k\bar{l}} \xi^k \bar{\xi}^l + w_{k\bar{l}} \left(\xi^k_{,\bar{i}} \bar{\xi}^l + \xi^k \bar{\xi}^l_{,\bar{i}} \right) \right]_{\bar{i}} \\ &= w_{k\bar{l}\bar{i}\bar{i}} \xi^k \bar{\xi}^l + w_{k\bar{l}} \left(\xi^k_{,\bar{i}} \bar{\xi}^l + \xi^k \bar{\xi}^l_{,\bar{i}} \right) + w_{k\bar{l}} \left(\xi^k_{,\bar{i}} \bar{\xi}^l + \xi^k \bar{\xi}^l_{,\bar{i}} \right) \\ &\quad + w_{k\bar{l}} \left(\xi^k_{,\bar{i}\bar{i}} \bar{\xi}^l + \xi^k_{,\bar{i}} \bar{\xi}^l_{,\bar{i}} + \xi^k_{,\bar{i}} \bar{\xi}^l_{,\bar{i}} + \xi^k \bar{\xi}^l_{,\bar{i}\bar{i}} \right) \\ &= w_{1\bar{1}\bar{i}\bar{i}} + w_{k\bar{l}} \xi^k_{,\bar{i}} \bar{\xi}^l_{,\bar{i}} + w_{1\bar{l}\bar{i}} \bar{\xi}^l_{,\bar{i}} + w_{k\bar{l}} \xi^k_{,\bar{i}} \bar{\xi}^l_{,\bar{i}} + w_{1\bar{l}\bar{i}} \bar{\xi}^l_{,\bar{i}} \\ &\quad + w_{1\bar{1}} (\xi^1_{,\bar{i}\bar{i}} + \bar{\xi}^1_{,\bar{i}\bar{i}}) + w_{k\bar{k}} (\xi^k_{,\bar{i}} \bar{\xi}^k_{,\bar{i}} + \xi^k_{,\bar{i}} \bar{\xi}^k_{,\bar{i}}) \\ &= w_{1\bar{1}\bar{i}\bar{i}} + 2 \sum_{k \neq 1} \text{Re}(w_{k\bar{1}} \xi^k_{,\bar{i}} \bar{\xi}^k_{,\bar{i}} + w_{1\bar{k}} \bar{\xi}^k_{,\bar{i}} \xi^k_{,\bar{i}}) + w_{1\bar{1}} (\xi^1_{,\bar{i}\bar{i}} + \bar{\xi}^1_{,\bar{i}\bar{i}}) + w_{k\bar{k}} (|\xi^k_{,\bar{i}}|^2 + |\bar{\xi}^k_{,\bar{i}}|^2). \end{aligned}$$

The last equality holds because we have used (4.2) and (4.5) and the fact

$$\begin{aligned} w_{k\bar{1}} \xi^k_{,\bar{i}} \bar{\xi}^l_{,\bar{i}} + w_{1\bar{l}} \bar{\xi}^l_{,\bar{i}} \xi^k_{,\bar{i}} &= 2\text{Re}(w_{k\bar{1}} \xi^k_{,\bar{i}} \bar{\xi}^l_{,\bar{i}}), \\ w_{1\bar{l}} \bar{\xi}^l_{,\bar{i}} \xi^k_{,\bar{i}} + w_{k\bar{1}} \xi^k_{,\bar{i}} \bar{\xi}^l_{,\bar{i}} &= 2\text{Re}(w_{1\bar{k}} \bar{\xi}^k_{,\bar{i}} \xi^k_{,\bar{i}}). \end{aligned}$$

We can also calculate

$$\begin{aligned}
\left(g^{p\bar{q}}w_{k\bar{q}}w_{p\bar{l}}\xi^k\bar{\xi}^l\right)_{\bar{i}\bar{i}} &= g^{p\bar{q}}\left(w_{k\bar{q}i}w_{p\bar{l}}\xi^k\bar{\xi}^l + w_{k\bar{q}}w_{p\bar{l}i}\xi^k\bar{\xi}^l + w_{k\bar{q}}w_{p\bar{l}}\xi^k\bar{\xi}^l + w_{k\bar{q}}w_{p\bar{l}}\xi^k\bar{\xi}^l\right)_{\bar{i}} \\
&= g^{p\bar{q}}\left(w_{k\bar{q}\bar{i}}w_{p\bar{l}}\xi^k\bar{\xi}^l + w_{k\bar{q}i}w_{p\bar{l}\bar{i}}\xi^k\bar{\xi}^l + w_{k\bar{q}i}w_{p\bar{l}}\xi^k\bar{\xi}^l + w_{k\bar{q}i}w_{p\bar{l}}\xi^k\bar{\xi}^l\right) \\
&\quad + g^{p\bar{q}}\left(w_{k\bar{q}\bar{i}}w_{p\bar{l}i}\xi^k\bar{\xi}^l + w_{k\bar{q}}w_{p\bar{l}\bar{i}}\xi^k\bar{\xi}^l + w_{k\bar{q}}w_{p\bar{l}i}\xi^k\bar{\xi}^l + w_{k\bar{q}}w_{p\bar{l}i}\xi^k\bar{\xi}^l\right) \\
&\quad + g^{p\bar{q}}\left(w_{k\bar{q}i}w_{p\bar{l}}\xi^k\bar{\xi}^l + w_{k\bar{q}}w_{p\bar{l}i}\xi^k\bar{\xi}^l + w_{k\bar{q}}w_{p\bar{l}}\xi^k\bar{\xi}^l + w_{k\bar{q}}w_{p\bar{l}}\xi^k\bar{\xi}^l\right) \\
&\quad + g^{p\bar{q}}\left(w_{k\bar{q}\bar{i}}w_{p\bar{l}}\xi^k\bar{\xi}^l + w_{k\bar{q}}w_{p\bar{l}\bar{i}}\xi^k\bar{\xi}^l + w_{k\bar{q}}w_{p\bar{l}}\xi^k\bar{\xi}^l + w_{k\bar{q}}w_{p\bar{l}}\xi^k\bar{\xi}^l\right) \\
&= w_{1\bar{1}\bar{i}}w_{1\bar{i}} + w_{1\bar{p}i}w_{p\bar{i}} + w_{k\bar{1}i}w_{1\bar{i}}\xi^k\bar{\xi}^l + w_{1\bar{p}i}w_{p\bar{i}}\xi^k\bar{\xi}^l \\
&\quad + w_{1\bar{p}i}w_{p\bar{i}} + w_{1\bar{1}}w_{1\bar{i}}\xi^k\bar{\xi}^l + w_{p\bar{p}}w_{p\bar{i}}\xi^k\bar{\xi}^l + w_{1\bar{1}}w_{1\bar{i}}\xi^k\bar{\xi}^l \\
&\quad + w_{k\bar{1}i}w_{1\bar{i}}\xi^k\bar{\xi}^l + w_{p\bar{p}}w_{p\bar{i}}\xi^k\bar{\xi}^l + w_{1\bar{1}}^2\xi^1\bar{\xi}^1 + w_{p\bar{p}}^2\xi^p\bar{\xi}^p \\
&\quad + w_{1\bar{p}i}w_{p\bar{i}}\xi^k\bar{\xi}^l + w_{1\bar{1}}w_{1\bar{i}}\xi^k\bar{\xi}^l + w_{p\bar{p}}^2\xi^p\bar{\xi}^p + w_{1\bar{1}}^2\xi^1\bar{\xi}^1 \\
&= 2w_{1\bar{1}}w_{1\bar{i}} + |w_{1\bar{p}i}|^2 + |w_{1\bar{p}i}|^2 + 2w_{1\bar{1}}\text{Re}(w_{p\bar{1}i}\xi^p\bar{\xi}^l + w_{p\bar{1}i}\xi^p\bar{\xi}^l) \\
&\quad + 2w_{p\bar{p}}\text{Re}(w_{1\bar{p}i}\xi^p\bar{\xi}^l + w_{p\bar{1}i}\xi^p\bar{\xi}^l) + w_{p\bar{p}}^2\left(|\xi^p\bar{\xi}^p|^2 + |\xi^p\bar{\xi}^p|^2\right) + w_{1\bar{1}}^2(\xi^1\bar{\xi}^1 + \xi^1\bar{\xi}^1)
\end{aligned}$$

Therefore we have simplify $Q_{\bar{i}\bar{i}}$ at x_0 as follows

$$\begin{aligned}
Q_{\bar{i}\bar{i}} &= (1 + 2c_0)\frac{w_{1\bar{1}\bar{i}}}{w_{1\bar{1}}} + \frac{c_0}{w_{1\bar{1}}^2} \sum_{p \neq 1} \left(|w_{1\bar{p}i}|^2 + |w_{1\bar{p}i}|^2\right) \\
&\quad - (1 + 2c_0)\frac{|w_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} + (**)_{\bar{i}\bar{i}} + \varphi_{\bar{i}\bar{i}} + \psi_{\bar{i}\bar{i}},
\end{aligned}$$

where $(**)_{\bar{i}\bar{i}}$ is given by

$$\begin{aligned}
(**)_{\bar{i}\bar{i}} &= \frac{2}{w_{1\bar{1}}} \sum_{k \neq 1} \text{Re}(w_{k\bar{1}i}\xi^k\bar{\xi}^l + w_{1\bar{k}i}\xi^k\bar{\xi}^l) + \xi^1\bar{\xi}^1 + \xi^1\bar{\xi}^1 + \frac{w_{k\bar{k}}}{w_{1\bar{1}}}(|\xi^k\bar{\xi}^k|^2 + |\xi^k\bar{\xi}^k|^2) \\
&\quad + \frac{2c_0}{w_{1\bar{1}}} \sum_{p \neq 1} \text{Re}(w_{p\bar{1}i}\xi^p\bar{\xi}^l + w_{p\bar{1}i}\xi^p\bar{\xi}^l) + \sum_{p \neq 1} \frac{2c_0w_{p\bar{p}}}{w_{1\bar{1}}^2} \text{Re}(w_{1\bar{p}i}\xi^p\bar{\xi}^l + w_{p\bar{1}i}\xi^p\bar{\xi}^l) \\
&\quad + \frac{2c_0w_{p\bar{p}}^2}{w_{1\bar{1}}^2} \left(|\xi^p\bar{\xi}^p|^2 + |\xi^p\bar{\xi}^p|^2\right) + c_0(\xi^1\bar{\xi}^1 + \xi^1\bar{\xi}^1).
\end{aligned}$$

For this term $(**)_{\bar{i}\bar{i}}$, we have the following estimate

$$(**)_{\bar{i}\bar{i}} \geq -\frac{c_0}{2w_{1\bar{1}}^2} \sum_{p \neq 1} \left(|w_{1\bar{p}i}|^2 + |w_{1\bar{p}i}|^2\right) - C,$$

where C is a positive constant depending only on (M, ω) .

Let

$$F(\omega_u) = (\sigma_k(\omega_u))^{1/k}.$$

We denote by

$$F^{i\bar{j}} = \frac{\partial F}{\partial w_{i\bar{j}}}, \quad F^{i\bar{j}, p\bar{q}} = \frac{\partial^2 F}{\partial w_{i\bar{j}} \partial w_{p\bar{q}}},$$

where $(w_u)_{i\bar{j}} = g_{i\bar{j}} + u_{i\bar{j}}$. Then, the positive definite matrix $(F^{i\bar{j}}(\omega_u))$ is diagonalized at the point x_0 . More precisely, we have

$$(4.8) \quad F^{i\bar{j}}(\omega_u) = \delta_{ij} F^{\bar{i}\bar{j}}(\omega_u) = \frac{1}{k} [\sigma_k(\lambda)]^{1/k-1} \sigma_{k-1}(\lambda|i) \delta_{ij}.$$

Furthermore, at x_0 ,

$$(4.9) \quad F^{i\bar{j}, p\bar{q}}(\omega_u) = \begin{cases} F^{\bar{i}\bar{j}, p\bar{p}}, & \text{if } i = j, p = q; \\ F^{i\bar{p}, p\bar{i}}, & \text{if } i = q, p = j, i \neq p; \\ 0, & \text{otherwise,} \end{cases}$$

in which

$$\begin{aligned} F^{\bar{i}\bar{j}, p\bar{p}} &= \frac{1}{k} [\sigma_k(\lambda)]^{1/k-1} (1 - \delta_{ip}) \sigma_{k-2}(\lambda|ip) \\ &\quad + \frac{1}{k} \left(\frac{1}{k} - 1 \right) [\sigma_k(\lambda)]^{1/k-2} \sigma_{k-1}(\lambda|i) \sigma_{k-1}(\lambda|p), \\ F^{i\bar{p}, p\bar{i}} &= -\frac{1}{k} [\sigma_k(\lambda)]^{1/k-1} \sigma_{k-2}(\lambda|ip). \end{aligned}$$

Here and in the follows, $\sigma_r(\lambda|i_1 \dots i_l)$, with i_1, \dots, i_l being distinct, stands for the r -th symmetric function with $\lambda_{i_1} = \dots = \lambda_{i_l} = 0$.

We have, in addition at x_0 ,

$$(4.10) \quad \sum_{i=1}^n F^{\bar{i}\bar{i}} w_{\bar{i}\bar{i}} = \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_i = \sigma_k^{1/k} = e^{\frac{f}{k}}.$$

Thus by maximum principal, we have

$$\begin{aligned}
 (4.11) \quad 0 &\geq F^{i\bar{j}}Q_{i\bar{j}} = F^{\bar{i}\bar{j}}Q_{\bar{i}\bar{j}} \\
 &\geq (1 + 2c_0) \sum_{i=1}^n \frac{F^{\bar{i}\bar{i}}u_{1\bar{i}\bar{i}}}{w_{1\bar{i}}} + \frac{c_0}{2} \sum_{i=1}^n \sum_{p \neq 1} \frac{F^{\bar{i}\bar{i}}|u_{1\bar{p}i}|^2}{w_{1\bar{i}}^2} \\
 &\quad - (1 + 2c_0) \sum_{i=1}^n \frac{F^{\bar{i}\bar{i}}|u_{1\bar{i}i}|^2}{w_{1\bar{i}}^2} + \psi' \sum_{i=1}^n F^{\bar{i}\bar{i}}u_{\bar{i}\bar{i}} + \psi'' \sum_{i=1}^n F^{\bar{i}\bar{i}}|u_i|^2 \\
 &\quad + \varphi'' \sum_{i=1}^n F^{\bar{i}\bar{i}}|\nabla u|_i^2 |\nabla u|_{\bar{i}}^2 + \varphi' \sum_{i,p=1}^n F^{\bar{i}\bar{i}}(|u_{p\bar{i}}|^2 + |u_{p\bar{i}}|^2) \\
 &\quad + \varphi' \sum_{i,p=1}^n F^{\bar{i}\bar{i}}(u_{p\bar{i}\bar{i}}u_{\bar{p}} + u_{\bar{p}\bar{i}\bar{i}}u_p) - C_1 \sum_{i=1}^n F^{\bar{i}\bar{i}} \\
 &:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8
 \end{aligned}$$

The equation can be written as

$$F(\omega_u) = e^{\frac{f}{k}} := h$$

Differentiate the above equation , we obtain

$$\begin{aligned}
 \sum_{i,j=1}^n F^{i\bar{j}}u_{i\bar{j}l} &= \nabla_l F = h_l, \\
 \sum_{i,j=1}^n F^{i\bar{j}}u_{i\bar{j}l\bar{m}} + \sum_{i,j,p,q=1}^n F^{i\bar{j},p\bar{q}}u_{i\bar{j}l}u_{p\bar{q}\bar{m}} &= h_{l\bar{m}}.
 \end{aligned}$$

and

$$\sum_{i=1}^n F^{\bar{i}\bar{i}}u_{\bar{i}\bar{i}1\bar{i}} = h_{1\bar{i}} - \sum_{i,j,p,q=1}^n F^{i\bar{j},p\bar{q}}u_{i\bar{j}1}u_{p\bar{q}\bar{i}}.$$

By commuting the covariant derivatives formula (2.10), we have

$$\begin{aligned}
 (4.12) \quad \sum_{i=1}^n F^{\bar{i}\bar{i}}u_{1\bar{i}\bar{i}} &= \sum_{i=1}^n F^{\bar{i}\bar{i}}u_{\bar{i}\bar{i}1\bar{i}} + \sum_{i=1}^n F^{\bar{i}\bar{i}} \left(u_{1\bar{i}} - \sum_{i=1}^n u_{\bar{i}\bar{i}} \right) R_{\bar{i}\bar{i}1\bar{i}} \\
 &\quad + \sum_{i=1}^n F^{\bar{i}\bar{i}} \left(\sum_{p=1}^n T_{1i}^p u_{p\bar{i}} + \sum_{q=1}^n \bar{T}_{1i}^q u_{1\bar{q}i} - \sum_{p=1}^n |T_{1i}^p|^2 u_{p\bar{p}} \right).
 \end{aligned}$$

Inserting (4.12) into the term I_1 , we have

$$\begin{aligned}
(4.13) \quad I_1 &= (1 + 2c_0) \sum_{i=1}^n \frac{F^{\bar{i}\bar{i}} u_{1\bar{i}\bar{i}}}{w_{1\bar{i}}} \\
&= (1 + 2c_0) \sum_{i=1}^n \frac{F^{\bar{i}\bar{i}} u_{\bar{i}\bar{i}1\bar{i}}}{w_{1\bar{i}}} + (1 + 2c_0) \sum_{i=1}^n \frac{F^{\bar{i}\bar{i}} (u_{1\bar{i}} - u_{\bar{i}\bar{i}}) R_{\bar{i}\bar{i}1\bar{i}}}{w_{1\bar{i}}} \\
&\quad + 2(1 + 2c_0) \sum_{i,p=1}^n F^{\bar{i}\bar{i}} \operatorname{Re} \left(\frac{T_{1i}^p u_{p\bar{i}\bar{i}}}{w_{1\bar{i}}} \right) - (1 + 2c_0) \sum_{i,p=1}^n F^{\bar{i}\bar{i}} \frac{|T_{1i}^p|^2 u_{p\bar{p}}}{w_{1\bar{i}}} \\
&= (1 + 2c_0) \frac{h_{1\bar{i}}}{w_{1\bar{i}}} - (1 + 2c_0) \sum_{i,j,p,q=1}^n \frac{F^{i\bar{j},p\bar{q}} u_{i\bar{j}1} u_{p\bar{q}\bar{i}}}{w_{1\bar{i}}} \\
&\quad + (1 + 2c_0) \sum_{i=1}^n \frac{F^{\bar{i}\bar{i}} (u_{1\bar{i}} - u_{\bar{i}\bar{i}}) R_{\bar{i}\bar{i}1\bar{i}}}{w_{1\bar{i}}} + 2(1 + 2c_0) \sum_i F^{\bar{i}\bar{i}} \operatorname{Re} \left(\frac{T_{1i}^1 u_{1\bar{i}\bar{i}}}{w_{1\bar{i}}} \right) \\
&\quad + 2(1 + 2c_0) \sum_{i=1}^n F^{\bar{i}\bar{i}} \operatorname{Re} \left(\sum_{p \neq 1} \frac{T_{1i}^p u_{p\bar{i}\bar{i}}}{w_{1\bar{i}}} \right) - (1 + 2c_0) \sum_{i,p=1}^n F^{\bar{i}\bar{i}} \frac{|T_{1i}^p|^2 u_{p\bar{p}}}{w_{1\bar{i}}} \\
&:= I_{11} + I_{12} + I_{13} + I_{14} + I_{15} + I_{16}.
\end{aligned}$$

Next we estimate each term of (1) as follows, firstly we have

$$I_{11} + I_{13} + I_{16} \geq -C_1 - 3(nC_2 + C_3) \sum_{i=1}^n F^{\bar{i}\bar{i}},$$

where we have supposed that $\sup_M |T|_g^2 \leq C_2$, $\sup_M |R| \leq C_3$.

Next we claim $I_{15} + I_2 \geq -18n^2 C_2 \sum_{i=1}^n F^{\bar{i}\bar{i}}$. In fact,

$$\begin{aligned}
I_{15} + I_{16} &= \frac{c_0}{2} \sum_{i=1}^n \sum_{p \neq 1} \frac{F^{\bar{i}\bar{i}} |u_{1\bar{p}i}|^2}{w_{1\bar{i}}^2} + 2(1 + 2c_0) \sum_{i=1}^n F^{\bar{i}\bar{i}} \operatorname{Re} \left(\sum_{p \neq 1} \frac{T_{1i}^p u_{p\bar{i}\bar{i}}}{w_{1\bar{i}}} \right) \\
&= \frac{c_0}{2} \sum_{i=1}^n F^{\bar{i}\bar{i}} \sum_{p \neq 1} \left| \frac{u_{1\bar{p}i}}{w_{1\bar{i}}} + \frac{2(1 + 2c_0)}{c_0} T_{1i}^p \right|^2 - \frac{2(1 + 2c_0)^2}{c_0} \sum_{i=1}^n \sum_{p \neq 1} F^{\bar{i}\bar{i}} |T_{1i}^p|^2 \\
&\geq -\frac{2(1 + 2c_0)^2}{c_0} \sum_{i=1}^n \sum_{p \neq 1} F^{\bar{i}\bar{i}} |T_{1i}^p|^2 \\
&\geq -18n^2 C_2 \sum_{i=1}^n F^{\bar{i}\bar{i}},
\end{aligned}$$

where we have used $\frac{1}{n^2} \leq c_0 \leq 1$ Thus, we obtain,

$$(4.14) \quad I_1 + I_2 \geq -(1 + 2c_0) \sum_{i,j,p,q=1}^n \frac{F^{i\bar{j},p\bar{q}} u_{i\bar{j}1} u_{p\bar{q}1}}{w_{1\bar{1}}} + 2(1 + 2c_0) \sum_i^n F^{i\bar{i}} \operatorname{Re} \left(\frac{T_{1i}^1 u_{1\bar{1}i}}{w_{1\bar{1}}} \right) \\ - (21n^2 C_2 + 3C_3) \sum_{i=1}^n F^{i\bar{i}} - C_1.$$

For terms $I_7 + I_8$, we claim

$$(4.15) \quad I_7 + I_8 \geq \frac{1}{2} \varphi' \sum_{i=1}^n F^{i\bar{i}} |u_{i\bar{i}}|_1^2 - (C_2 + C_3) \sum_{i=1}^n F^{i\bar{i}} - C_1.$$

Indeed, by the covariant derivatives' commutation formula (2.9) in section 2, we have

$$u_{p\bar{i}} = u_{i\bar{p}} + T_{pi}^i u_{i\bar{i}} + u_q R_{i\bar{p}q}^i, u_{\bar{p}i} = u_{i\bar{p}} = u_{i\bar{p}} - \overline{T_{ip}^i} u_{i\bar{i}}.$$

Then we have

$$\sum_{i=1}^n F^{i\bar{i}} u_{p\bar{i}} = \sum_{i=1}^n F^{i\bar{i}} u_{i\bar{p}} + \sum_{i=1}^n F^{i\bar{i}} (T_{pi}^i u_{i\bar{i}} + u_q R_{i\bar{p}q}^i) = F_p + \sum_{i=1}^n F^{i\bar{i}} (T_{pi}^i u_{i\bar{i}} + u_q R_{i\bar{p}q}^i) \\ \sum_{i=1}^n F^{i\bar{i}} u_{\bar{p}i} = \sum_{i=1}^n F^{i\bar{i}} u_{i\bar{p}} + \sum_{i=1}^n F^{i\bar{i}} \overline{T_{ip}^i} u_{i\bar{i}} = F_{\bar{p}} + \sum_{i=1}^n F^{i\bar{i}} \overline{T_{ip}^i} u_{i\bar{i}}$$

Inserting the above formula into the term (8), we obtain

$$(4.16) \quad I_8 = \varphi' \sum_{i,p=1}^n F^{i\bar{i}} (u_{p\bar{i}} u_{\bar{p}i} + u_{\bar{p}i} u_p) \\ = \varphi' \sum_{p=1}^n u_{\bar{p}} \left[F_p + \sum_{i=1}^n F^{i\bar{i}} (T_{pi}^i u_{i\bar{i}} + u_q R_{i\bar{p}q}^i) \right] + \varphi' \sum_{p=1}^n u_p \left[h_{\bar{p}} - \sum_{i=1}^n F^{i\bar{i}} \overline{T_{ip}^i} u_{i\bar{i}} \right] \\ = 2\varphi' \sum_{i,p=1}^n F^{i\bar{i}} u_{i\bar{i}} \operatorname{Re} (u_{\bar{p}} T_{pi}^i) + \varphi' \sum_{p=1}^n \left[2\operatorname{Re} (u_{\bar{p}} h_p) + \sum_{i,q=1}^n u_{\bar{p}} u_q F^{i\bar{i}} R_{i\bar{p}q}^i \right] \\ = I_{81} + I_{82}.$$

For the term I_{82} , we have

$$I_{82} \geq -C_3 \sum_{i=1}^n F^{i\bar{i}} - C_1.$$

For the term I_{81} , we obtain

$$\begin{aligned}
I_{81} + I_7 &= 2\varphi' \sum_{i,p=1}^n F^{\bar{i}\bar{i}} u_{\bar{i}\bar{i}} \operatorname{Re}(u_{\bar{p}} T_{pi}^i) + \varphi' \sum_{i,p=1}^n F^{\bar{i}\bar{i}} (|u_{\bar{p}\bar{i}}|^2 + |u_{pi}|^2) \\
&\geq \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \left[|u_{\bar{i}\bar{i}}|^2 + 2u_{\bar{i}\bar{i}} \operatorname{Re} \left(\sum_{p=1}^n u_{\bar{p}} T_{pi}^i \right) \right] \\
&= \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \left| \frac{u_{\bar{i}\bar{i}}}{2} + 2 \sum_{p=1}^n u_p \overline{T_{pi}^i} \right|^2 + \frac{3}{4} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} |u_{\bar{i}\bar{i}}|^2 - 4\varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \left| \sum_{p=1}^n u_p \overline{T_{pi}^i} \right|^2 \\
&\geq \frac{1}{2} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} |u_{\bar{i}\bar{i}}|^2 - C_2 \sum_{i=1}^n F^{\bar{i}\bar{i}}.
\end{aligned}$$

Thus we have proved the above claim (4.15).

Moreover, apply (4.10) to obtain

$$\psi' \sum_{i=1}^n F^{\bar{i}\bar{i}} u_{\bar{i}\bar{i}} = \psi' \sum_{i=1}^n F^{\bar{i}\bar{i}} (\lambda_i - 1) = \psi' h - \psi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \geq -2(C_0 + 1) \sup_M e^{\frac{f}{k}} + \psi' \sum_{i=1}^n F^{\bar{i}\bar{i}}$$

Similarly,

$$\begin{aligned}
\frac{1}{2} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} |u_{\bar{i}\bar{i}}|^2 &= \frac{1}{2} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} (\lambda_i - 1)^2 \\
&= \frac{1}{2} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_i^2 - \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_i + \frac{1}{2} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \\
&= \frac{1}{2} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_i^2 - \varphi' h + \frac{1}{2} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \\
&\geq \frac{1}{2} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_i^2 - \frac{1}{2} \sup_M e^{\frac{f}{k}} + \frac{1}{2} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}}
\end{aligned}$$

Inserting these terms into (4.11), we obtain

(4.17)

$$0 \geq F^{\bar{i}\bar{i}} Q_{\bar{i}\bar{i}} \geq -(1+2c_0) \sum_{i,j,p,q=1}^n \frac{F^{i\bar{j},p\bar{q}} u_{i\bar{j}1} u_{p\bar{q}1}}{w_{1\bar{1}}} + 2(1+2c_0) \sum_{i=1}^n F^{\bar{i}\bar{i}} \operatorname{Re} \left(\frac{T_{1i}^1 u_{1\bar{i}}}{w_{1\bar{1}}} \right) - (1+2c_0) \sum_{i=1}^n \frac{F^{\bar{i}\bar{i}} |u_{1\bar{i}}|^2}{w_{1\bar{1}}^2}$$

(4.18)

$$\begin{aligned} & + \varphi'' \sum_{i=1}^n F^{\bar{i}\bar{i}} |\nabla u|_i^2 |\nabla u|_{\bar{i}}^2 + \psi'' \sum_{i=1}^n F^{\bar{i}\bar{i}} |u_i|^2 + \frac{1}{2} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_i^2 \\ & + \left(-\psi' + \frac{1}{2} \varphi' - 22n^2 C_2 - 4C_3 \right) \sum_{i=1}^n F^{\bar{i}\bar{i}} - C_1 \\ & = A_1 + A_2 + A_3 \\ & + A_4 + A_5 + A_6 + \left(-\psi' + \frac{1}{2} \varphi' - 22n^2 C_2 - 4C_3 \right) \sum_{i=1}^n F^{\bar{i}\bar{i}} - C_1, \end{aligned}$$

where C_1 is a positive constant depending only on C_0 , $\sup e^{\frac{f}{k}}$, and $\sup \left| \nabla \left(e^{\frac{f}{k}} \right) \right|^2$, and $\sup \left| \partial \bar{\partial} \left(e^{\frac{f}{k}} \right) \right|$.

Let $\varepsilon = \frac{\delta}{4} \leq \frac{1}{16}$ and $\delta = \frac{1}{2A+1}$, where $A = 2L(C_0 + 1)$ and $C_0 = 31n^2 C_2 + 4C_3$. We divide two cases to drive the estimate, which is similar as [4].

Case1: $\lambda_n < -\varepsilon \lambda_1$.

By the first derivative's condition (4.7), we have

$$\begin{aligned} -(1+2c_0)^2 \left| \frac{u_{1\bar{i}}}{w_{1\bar{1}}} \right|^2 &= -|\varphi' |\nabla u|_i^2 + \psi' u_i|^2 \geq -2(\varphi')^2 |\nabla u|_i^2 |\nabla u|_{\bar{i}}^2 - 2(\psi')^2 |u_i|^2 \\ &= -\varphi'' |\nabla u|_i^2 |\nabla u|_{\bar{i}}^2 - 2(\psi')^2 |u_i|^2, 1 \leq i \leq n \end{aligned}$$

$$\begin{aligned} A_2 &= 2(1+2c_0) \sum_{i \neq 1} F^{\bar{i}\bar{i}} \operatorname{Re} \left(\frac{T_{1i}^1 u_{1\bar{i}}}{w_{1\bar{1}}} \right) \\ &\geq -c_0 \sum_{i \neq 1} F^{\bar{i}\bar{i}} \left| \frac{u_{1\bar{i}}}{w_{1\bar{1}}} \right|^2 - \frac{(1+2c_0)^2}{c_0} \sum_{i \neq 1} F^{\bar{i}\bar{i}} |T_{1i}^1|^2 \\ &\geq -c_0 \sum_{i \neq 1} F^{\bar{i}\bar{i}} \left| \frac{u_{1\bar{i}}}{w_{1\bar{1}}} \right|^2 - 9n^2 C_2 \sum_{i \neq 1} F^{\bar{i}\bar{i}} |T_{1i}^1|^2 \end{aligned}$$

Thus

$$\begin{aligned}
A_2 + A_3 &\geq -(1 + 3c_0) \sum_{i=1}^n \frac{F^{\bar{i}\bar{i}} |u_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} - 9n^2 C_2 \sum_{i \neq 1} F^{\bar{i}\bar{i}} |T_{1i}^1|^2 \\
&\geq -(1 + 2c_0)^2 \sum_{i=1}^n \frac{F^{\bar{i}\bar{i}} |u_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} - 9n^2 C_2 \sum_{i=1}^n F^{\bar{i}\bar{i}} \\
&= -A_4 - 2(\psi')^2 \sum_{i=1}^n F^{\bar{i}\bar{i}} |u_i|^2 - 9n^2 C_2 \sum_{i=1}^n F^{\bar{i}\bar{i}}.
\end{aligned}$$

We therefore obtain

$$(4.19) \quad A_2 + A_3 + A_4 \geq -2(\psi')^2 \sum_{i=1}^n F^{\bar{i}\bar{i}} |u_i|^2 - 9n^2 C_2 \sum_{i=1}^n F^{\bar{i}\bar{i}}.$$

Using the following inequality

$$\sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_i^2 \geq F^{n\bar{n}} \lambda_n^2 > \varepsilon^2 F^{n\bar{n}} \lambda_1^2 \geq \frac{\varepsilon^2}{n} \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_1^2.$$

Therefore, we have

$$(4.20) \quad A_6 = \frac{1}{2} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_i^2 \geq \frac{\varepsilon^2}{2n} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_1^2$$

Combining (4.17) and (4.19) (4.20), we obtain

$$\begin{aligned}
0 &\geq \sum_{i=1}^n F^{\bar{i}\bar{i}} Q_{\bar{i}\bar{i}} \geq \frac{\varepsilon^2}{2n} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_1^2 - 2(\psi')^2 \sum_{i=1}^n F^{\bar{i}\bar{i}} |u_i|^2 \\
&\quad + \left(-\psi' + \frac{1}{2} \varphi' - 31n^2 C_2 - 4C_3 \right) \sum_{i=1}^n F^{\bar{i}\bar{i}} - C_1 \\
&\geq \left(\frac{\varepsilon^2}{8nK} \lambda_1^2 - 8K(C_0 + 1)^2 \right) \sum_{i=1}^n F^{\bar{i}\bar{i}} - C_1 \\
&\geq \frac{\varepsilon^2}{8nK} \lambda_1^2 - 8K(C_0 + 1)^2 - C_1,
\end{aligned}$$

where we have used the fact that $\sum_{i=1}^n F^{\bar{i}\bar{i}} \geq 1$, which follows from Newton-Maclaurin's inequality and the fact that .

Hence, we obtain the estimates

$$\lambda_1 \leq 8\sqrt{2}(2A + 1) \sqrt{nK(8K(C_0 + 1)^2 + C_1)} \leq CK.$$

Case2: $\lambda_n > -\varepsilon\lambda_1$.

Let

$$I = \left\{ i \in \{1, \dots, n\} \mid \sigma_{k-1}(\lambda|i) \geq \varepsilon^{-1} \sigma_{k-1}(\lambda|1) \right\}$$

Obviously, $1 \notin I$ and $i \in I$ if and only if

$$F^{\bar{i}\bar{i}} > \varepsilon^{-1} F^{1\bar{1}}$$

We first treat those indices which are not in I : by the first derivative's condition (4.7), we have

$$\begin{aligned} & -(1 + 2c_0) \sum_{i \notin I} \frac{F^{\bar{i}\bar{i}} |u_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} + 2(1 + 2c_0) \sum_{i \notin I} F^{\bar{i}\bar{i}} \operatorname{Re} \frac{T_{1i}^1 u_{1\bar{1}\bar{i}}}{w_{1\bar{1}}} \\ & \geq -(1 + 2c_0)^2 \sum_{i \notin I} \frac{F^{\bar{i}\bar{i}} |u_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} - \frac{(1 + 2c_0)^2}{c_0} \sum_{i \notin I} F^{\bar{i}\bar{i}} |T_{1i}^1|^2 \\ & = -\varphi'' \sum_{i \notin I} F^{\bar{i}\bar{i}} |\nabla u|_i^2 |\nabla u|_{\bar{i}}^2 - 2(\psi')^2 \sum_{i \notin I} F^{\bar{i}\bar{i}} |u_i|^2 - 9n^2 C_2 \sum_{i \notin I} F^{\bar{i}\bar{i}} |T_{1i}^1|^2 \\ & \geq -\varphi'' \sum_{i \notin I} F^{\bar{i}\bar{i}} |\nabla u|_i^2 |\nabla u|_{\bar{i}}^2 - 2\varepsilon^{-1} K(\psi')^2 F^{1\bar{1}} - 9n^2 C_2 \sum_{i=1}^n F^{\bar{i}\bar{i}} \end{aligned}$$

Substitute the above inequality into (4.17)

(4.21)

$$\begin{aligned} 0 \geq F^{\bar{i}\bar{i}} Q_{\bar{i}\bar{i}} & \geq -(1 + 2c_0) \sum_{i,j,p,q=1}^n \frac{F^{i\bar{j},p\bar{q}} u_{i\bar{j}1} u_{p\bar{q}\bar{1}}}{w_{1\bar{1}}} + 2(1 + 2c_0) \sum_{i \in I} F^{\bar{i}\bar{i}} \operatorname{Re} \left(\frac{T_{1i}^1 u_{1\bar{1}\bar{i}}}{w_{1\bar{1}}} \right) \\ & - (1 + 2c_0) \sum_{i \in I} \frac{F^{\bar{i}\bar{i}} |u_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} + \varphi'' \sum_{i \in I} F^{\bar{i}\bar{i}} |\nabla u|_i^2 |\nabla u|_{\bar{i}}^2 + \psi'' \sum_{i=1}^n F^{\bar{i}\bar{i}} |u_i|^2 \\ & + \frac{1}{2} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_i^2 - 2\varepsilon^{-1} K(\psi')^2 F^{1\bar{1}} + \left(-\psi' + \frac{1}{2} \varphi' - 31n^2 C_2 - 4C_3 \right) \sum_{i=1}^n F^{\bar{i}\bar{i}} - C_1 \\ & = B_1 + B_2 + B_3 + B_4 + B_5 \\ & + B_6 + B_7 + B_8 \end{aligned}$$

Firstly, we have

$$B_6 + B_7 = \frac{1}{2} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_i^2 - 2\varepsilon^{-1} K(\psi')^2 F^{1\bar{1}} \geq \frac{1}{4} \varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_i^2,$$

where we have assumed $\frac{1}{4} \varphi' F^{1\bar{1}} \lambda_1^2 \geq 2\varepsilon^{-1} K(\psi')^2 F^{1\bar{1}}$ otherwise we have $\frac{1}{4} \varphi' F^{1\bar{1}} \lambda_1^2 \leq 2\varepsilon^{-1} K(\psi')^2 F^{1\bar{1}}$ i.e. $\lambda_1 \leq CK$ and the estimate is done.

We next use the first term B_1 to cancel the other terms containing the third derivatives of u . By the same proof as in [4] P559, we have

$$\lambda_1 \sigma_{k-2}(\lambda | 1i) \geq (1 - 2\varepsilon) \sigma_{k-1}(\lambda | i), \text{ for } i \in I.$$

Thus

$$-\lambda_1 F^{i\bar{1}, 1\bar{i}} = \frac{F^{1-k}}{k} \lambda_1 \sigma_{k-2}(\lambda | 1i) \geq \frac{F^{1-k}}{k} (1 - 2\varepsilon) \sigma_{k-1}(\lambda | i) = (1 - 2\varepsilon) F^{\bar{i}\bar{i}}$$

Since

$$u_{i\bar{1}1} = u_{1\bar{i}i} - T_{1i}^1(\lambda_1 - 1)$$

Therefore

$$\begin{aligned} B_1 &= -\frac{1 + 2c_0}{\lambda_1} \sum_{i,j,p,q=1}^n F^{i\bar{j}, p\bar{q}} u_{i\bar{j}1} u_{p\bar{q}\bar{1}} \geq -\frac{1 + 2c_0}{\lambda_1^2} \sum_{i \in I} \lambda_1 F^{i\bar{1}, 1\bar{i}} u_{i\bar{1}1} u_{1\bar{i}\bar{1}} \\ &\geq \frac{1 + 2c_0}{\lambda_1^2} (1 - 2\varepsilon) \sum_{i \in I} F^{\bar{i}\bar{i}} |u_{1\bar{i}i} - T_{1i}^1(\lambda_1 - 1)|^2 \end{aligned}$$

$$B_2 = \frac{2(1 + 2c_0)}{\lambda_1} \sum_{i \in I} F^{\bar{i}\bar{i}} \operatorname{Re}(T_{1i}^1 u_{1\bar{i}\bar{1}})$$

From the first derivative's condition (4.7), we have

$$\begin{aligned} B_4 &= \varphi'' \sum_{i \in I} F^{\bar{i}\bar{i}} |\nabla u|_i^2 |\nabla u|_{\bar{i}}^2 = 2 \sum_{i \in I} F^{\bar{i}\bar{i}} \left| (1 + 2c_0) \frac{u_{1\bar{i}\bar{1}}}{w_{1\bar{i}}} + \psi' u_i \right|^2 \\ &\geq 2(1 + 2c_0)^2 \delta \sum_{i \in I} F^{\bar{i}\bar{i}} \frac{|u_{1\bar{i}\bar{1}}|^2}{w_{1\bar{i}}^2} - \frac{2\delta}{1 - \delta} (\psi')^2 \sum_{i \in I} F^{\bar{i}\bar{i}} |u_i|^2 \\ &\geq 2(1 + 2c_0)^2 \delta \sum_{i \in I} F^{\bar{i}\bar{i}} \frac{|u_{1\bar{i}\bar{1}}|^2}{w_{1\bar{i}}^2} - B_5, \end{aligned}$$

where we have used $\frac{2\delta}{1-\delta} (\psi')^2 = \psi''$ by our choosing $\delta = \frac{1}{2A+1}$.

Thus we have

$$B_3 + B_4 + B_5 \geq -(1 + 2c_0) \frac{[1 - 2(1 + 2c_0)\delta]}{\lambda_1^2} \sum_{i \in I} F^{\bar{i}\bar{i}} |u_{1\bar{i}\bar{1}}|^2.$$

Therefore,

$$\begin{aligned}
& B_1 + B_2 + B_3 + B_4 + B_5 \\
& \geq \frac{1+2c_0}{\lambda_1^2} (1-2\varepsilon) \sum_{i \in I} F^{\bar{i}\bar{i}} |u_{1\bar{i}i} - T_{1i}^1 (\lambda_1 - 1)|^2 - (1+2c_0) \frac{[1-2(1+2c_0)\delta]}{\lambda_1^2} \sum_{i \in I} F^{\bar{i}\bar{i}} |u_{1\bar{i}i}|^2 \\
& \quad + \frac{2(1+2c_0)}{\lambda_1^2} \sum_{i \in I} F^{\bar{i}\bar{i}} \operatorname{Re}(\lambda_1 T_{1i}^1 u_{1\bar{i}i}) \\
& = \frac{1+2c_0}{\lambda_1^2} \sum_{i \in I} F^{\bar{i}\bar{i}} \left\{ (1-2\varepsilon) |u_{1\bar{i}i} - T_{1i}^1 (\lambda_1 - 1)|^2 - (1-2(1+2c_0)\delta) |u_{1\bar{i}i}|^2 + 2\operatorname{Re}(\lambda_1 T_{1i}^1 u_{1\bar{i}i}) \right\} \\
& = \frac{1+2c_0}{\lambda_1^2} \sum_{i \in I} F^{\bar{i}\bar{i}} \left\{ (2(1+2c_0)\delta - 2\varepsilon) |u_{1\bar{i}i}|^2 + 2[2\varepsilon(\lambda_1 - 1) + 1] \operatorname{Re}(T_{1i}^1 u_{1\bar{i}i}) + (1-2\varepsilon)(\lambda_1 - 1)^2 |T_{1i}^1|^2 \right\} \\
& \geq 0,
\end{aligned}$$

where the last inequality holds if we choose $\varepsilon = \frac{\delta}{4} \leq \frac{1}{16}$. In fact,

$$\begin{aligned}
\Delta &= B^2 - 4AC = 4[2\varepsilon(\lambda_1 - 1) + 1]^2 - 4(1-2\varepsilon)(\lambda_1 - 1)^2(2(1+2c_0)\delta - 2\varepsilon) \\
&\leq 36\varepsilon^2(\lambda_1 - 1)^2 - 4(1-2\varepsilon)(\lambda_1 - 1)^2(2(1+2c_0)\delta - 2\varepsilon) \\
&\leq 4(\lambda_1 - 1)^2(9\varepsilon^2 - 2(1-2\varepsilon)((1+2c_0)\delta) + 2\varepsilon(1-2\varepsilon)) \\
&\leq 4(\lambda_1 - 1)^2(5\varepsilon^2 + 2\varepsilon - \delta) \\
&\leq 4(\lambda_1 - 1)^2(4\varepsilon - \delta) \\
&= 0.
\end{aligned}$$

Thus we finally obtain

$$\begin{aligned}
0 &\geq \frac{1}{4}\varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} |u_{\bar{i}i}|^2 + \left(-\psi' + \frac{1}{2}\varphi' - C_2 - C_3\right) \sum_{i=1}^n F^{\bar{i}\bar{i}} - C_1 \\
&= \left(-\psi' + \frac{1}{2}\varphi' - C_2 - C_3\right) \sum_{i=1}^n F^{\bar{i}\bar{i}} + \frac{1}{2}\varphi' \sum_{i=1}^n F^{\bar{i}\bar{i}} |u_{\bar{i}i}|^2 - C_1 \\
&\geq \sum_{i=1}^n F^{\bar{i}\bar{i}} + \frac{1}{16K} \sum_{i=1}^n F^{\bar{i}\bar{i}} \lambda_i^2 - C_1
\end{aligned}$$

, where we have used $-\psi' \geq C_0 + 1$ by choosing $C_0 = 31n^2C_2 + 4C_3$.

In particular, we have $\sum_{i=1}^n F^{\bar{i}\bar{i}} \leq C$. By lemma 2.2 in [4] we have $F^{1\bar{1}} \geq \frac{c(n,k)}{C_1^{k-1}}$, where $c(n,k)$ is a positive constant depending only on n and k .

Therefore, we get the desired estimate:

$$\lambda_1 \leq \frac{4C_1^{\frac{k}{2}}}{c(n, k)^{\frac{1}{2}}} \sqrt{K},$$

where C_1 is given in (4.17). □

REFERENCES

- [1] Cherrier P. *équations de Monge-Ampère sur les variétés hermitiennes compactes*[J]. Bulletin des sciences mathématiques, 1987, 111(4): 343-385.
- [2] Chou K S, Wang X J. *A variational theory of the Hessian equation*[J]. Communications on Pure and Applied Mathematics, 2001, 54(9): 1029-1064.
- [3] Dinew S, Kolodziej S. *Liouville and Calabi-Yau type theorems for complex Hessian equations*[J]. arXiv preprint arXiv:1203.3995, 2012.
- [4] Hou Z, Ma X N, Wu D. *A second order estimate for complex Hessian equations on a compact Kähler manifold*[J]. Mathematical research letters, 2010, 17(2): 547-562.
- [5] Guan B, Li Q. *Complex Monge-Ampère equations and totally real submanifolds*[J]. Advances in Mathematics, 2010, 225(3): 1185-1223.
- [6] Li Y. *A priori estimates for Donaldson's equation over compact Hermitian manifolds*[J]. Calculus of Variations and Partial Differential Equations, 2012: 1-16.
- [7] Lin M, Trudinger N S. *On some inequalities for elementary symmetric functions*[J]. Bulletin of the Australian Mathematical Society, 1994, 50(02): 317-326.
- [8] Sun W. *On uniform estimate of complex elliptic equations on closed Hermitian manifolds*[J]. arXiv preprint arXiv:1412.5001, 2014.
- [9] Weinkove B. *Convergence of the J-flow on Kähler Surfaces*[J]. Communications in analysis and geometry, 2004, 12(3): 949-965.
- [10] Tosatti V, Weinkove B. *Estimates for the complex Monge-Ampère equation on Hermitian and balanced manifolds*[J]. Asian Journal of Mathematics, 2010, 14(1): 19-40.
- [11] Tosatti V, Weinkove B. *The complex Monge-Ampère equation on compact Hermitian manifolds*[J]. Journal of the American Mathematical Society, 2010, 23(4): 1187-1195.
- [12] Tosatti V, Weinkove B. *On the evolution of a Hermitian metric by its Chern-Ricci form*[J]. preprint arXiv:1201.0312, 2011.
- [13] Tosatti V, Weinkove B. *The Monge-Ampère equation for $(n - 1)$ -plurisubharmonic functions on a compact Kähler manifold*[J]. preprint arXiv:1305.7511, 2013.
- [14] Tosatti V, Weinkove B. *Hermitian metrics, $(n - 1, n - 1)$ forms and Monge-Ampère equations*[J]. preprint arXiv:1310.6326, 2013.
- [15] Wang X J. *The k -Hessian equation*, Lecture Notes in Mathematics, Springer Berlin/Heidelberg, Volume **1977**, 2009, 177-252.
- [16] Yau S T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*[J]. Communications on pure and applied mathematics, 1978, 31(3): 339-411.
- [17] Zhang X. *A priori estimates for complex Monge-Ampère equation on Hermitian manifolds*[J]. International Mathematics Research Notices, 2010, 2010(19): 3814-3836.

UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, 230026, ANHUI PROVINCE, CHINA
E-mail address: dekzhang@mail.ustc.edu.cn